Lesson 1: Decimal Expansions of Rational Numbers

For Students

Performance Expectations (CCSS)

This lesson addresses the following Common Core State Standard (CCSS) for Grade 8:

- 8.NS.A.1: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion that repeats eventually into a rational number.

You learned previously that any number that can be written as the quotient of two integers is called a rational number. In this lesson, you are going to study the relationship between rational numbers and their decimal expansions.

Terminating decimal expansions

Some decimals, such as 0.57, are **terminating**. This means that they have a finite number of digits. A terminating decimal can be written as the quotient of two integers.

$$0.57 = \frac{57}{\square}$$

This demonstrates that 0.57 is a rational number.

Example 1

Show that –1.375 is a rational number.

Solution:

The decimal goes to the thousandths place, so you can use the denominator 1000.

$$-1.375 = \frac{-1375}{1000}$$

This is the quotient of two _____________, so it is rational.

Note that in the last example, the fraction can be reduced to $\frac{-11}{8}$. (This is not needed for the demonstration that the number is rational.) It is important to understand that the preceding numbers are all rational regardless of the form in which they are written. They were rewritten in fraction form merely to verify this property.

You can convert a fraction, such as $\frac{7}{40}$, to a decimal (0.175) by doing long division.

$$\begin{array}{r} 0.175 \\ 40\overline{)\;7.000} \\ -\underline{4.000} \\ 3.000 \\ -\underline{2.800} \\ 0.200 \\ -\underline{0.200} \\ 0 \end{array}$$

Not all fractions result in terminating decimals. As seen on the previous page, a terminating decimal can be written as a fraction whose denominator is a power of 10. That fraction may or may not reduce. In any case, the reduced form of the fraction has a denominator that is a divisor of a power of 10. Another way to say this is that the only prime divisors of the denominator of this reduced fraction are 2 and 5 (the only prime factors of 10). In the examples shown thus far, the denominators of the reduced fractions were $100 = 2^2 \cdot 5^2$, $8 = 2^3$, and $40 = 2^{\square} \cdot 5^{\square}$.

Conversely, if you start with a fraction whose prime divisors are only 2 and 5, you can multiply the numerator and denominator by a power of 2 or 5 to get equal powers (if you start with $2^3 \cdot 5$, say, you can multiply by 5^2). This yields a denominator that is a power of 10, and you can rewrite this fraction as a terminating decimal.

In summary, we have the following:

A fraction written in reduced form will have a terminating decimal if and only if the prime factors of the denominator are 2 and 5.

Note: This does **not** apply to a fraction that is not in reduced form. For example, you could rewrite $\frac{7}{40}$ as $\frac{\square}{280}$. This fraction's denominator has a prime factor of 7 and has a terminating decimal.

Repeating decimal expansions

If you convert a reduced fraction whose denominator has a prime factor other than 2 or 5, you will get a non-terminating decimal.

Let us do this with $\frac{5}{11}$:

$$\begin{array}{r} 0.454 \\ 11\overline{)\ 5.000} \\ -\underline{4.400} \\ 0.600 \\ -\underline{0.550} \\ 0.050 \\ -\underline{0.044} \\ 6 \end{array}$$

Notice that after the second subtraction there was a remainder of 5, leading to the same division as in the first step. Naturally, this led to the same remainder of 6 obtained after the first subtraction. So the fourth division will be the same as the second division. Then the fifth division will be the same as the ___________, and so on. This means that the digits 4 and 5 in the decimal form of the quotient will just keep repeating.

$$\frac{5}{11} = 0.45454545...$$

The ellipsis (the series of dots) is used to mean that the pattern continues as shown. There is a more precise notation to indicate this, however. Whenever digits in a decimal form a repeating pattern, an overbar can be drawn above the digits that form one whole pattern to indicate the repetition. For this example, we have:

$$\frac{5}{11} = 0.\overline{45}$$

Example 2

What is $\frac{1}{37}$ written in decimal form?

Solution:

Long division will at first yield a quotient of 0.027 with a remainder of ___, and then the pattern will repeat. So the quotient can be written as 0.027027027… or $0.\overline{027}$.

In this example, the digit 0 had to be under the overbar, as that digit is part of the repeating pattern. If you wrote $0.0\overline{27}$ instead, that would represent the fraction $\frac{3}{110}$.

When you divide any whole number by any nonzero whole number, you will get a **repeating decimal expansion** (the "repeating" part may just be zeros, in which case you do not write them and you have a terminating decimal), as shown in the examples above.

Why does this happen?

Look back at the long division by 11. The remainders after each step were 6, 5, and then 6 again. Once a remainder repeats, you will then repeat the same divisions and get the same quotients (digits in the decimal expansion). So you have a repeating pattern.

Why do the remainders have to repeat?

When you divide by 11, for example, there are only ____ different possible remainders. So if you performed long division, once you get to the 12th division, you have 12 remainders but they can be at most ____ different possible numbers. So at least two of the remainders must be the same. (In that example, it took only 3 divisions to get a repeat.)

Likewise, when you divide by n, there are n possible remainders, and this implies that a remainder must repeat by the n+1st division (if not sooner).

Now suppose that you have a repeating decimal. How do you convert it to a fraction? You cannot do what you did for terminating decimals: count the number of places to find the correct power of 10 to be the denominator, because there are an infinite number of decimal places.

You can convert $0.\overline{1234}$ to fraction form using some algebra. Call this number x.

$x = 0.123412341234...$	
$10{,}000x = 1234.123412341234...$	Multiply by 10,000.
$-(\underline{x = \quad 0.123412341234...})$	Subtract the original number.
$\boxed{}\,x = 1234$	The decimal parts cancel.
$x = \dfrac{1234}{9{,}999}$	Solve for x.

You can have a repeating decimal that starts with a non-repeating part.

Example 3

Convert $0.2\overline{15}$ into fraction form.

Solution:

Let $x = 0.2\overline{15}$. To obtain just the repeating part to the right of the decimal point, you can multiply this by 10 and by 1000, and then subtract:

$$1000x = 215.151515...$$
$$-(\underline{10x = \quad 2.151515...})$$
$$\boxed{}\,x = 213$$
$$x = \frac{213}{990}$$

Irrational numbers

It follows from all of the preceding explanations that any rational number either has a terminating or a repeating decimal expansion. The converse is also true: any number that has either a terminating or repeating decimal expansion is a rational number.

There are many numbers that cannot be written as the quotient of two integers, such as $\sqrt{2}$ and π. These are known as **irrational numbers**. (The enrichment material has a proof that $\sqrt{2}$ is irrational. The proof that π is irrational would be covered in a college-level mathematics course; its irrationality is assumed without proof here.) It follows from the preceding paragraph that the decimal expansions of irrational numbers must be non-terminating and non-repeating. The converse is also true: any non-terminating and non-repeating decimal must be irrational.

There are infinitely many irrational numbers. But unlike rational numbers, which can be written in the manner shown in this lesson, irrational numbers by definition cannot be written in this form. So they need to be described indirectly. You have been given definitions for $\sqrt{2}$ and π, so you know what these symbols mean, but it takes some work to arrive at the (approximate) decimal representations 1.414… and 3.1415… The ellipses do not tell you what the rest of the digits are.

Exercises for lesson 1

Write the decimals as fractions to show that they are rational.

1. 0.42   2. 0.967   3. 1.0523

A reduced fraction has the given number as a denominator.
Will the fraction be terminating or non-terminating?

4. 250   5. 30   6. 88

Write the fractions as decimals. Use an overbar, if necessary.

7. $\frac{2}{9}$ 8. $\frac{1}{16}$ 9. $\frac{8}{33}$ 10. $\frac{1}{125}$ 11. $\frac{31}{303}$ 12. $\frac{503}{3330}$

Write the decimals as fractions.

13. $0.\overline{7}$ 14. $0.\overline{61}$ 15. $0.\overline{940}$ 16. $0.3\overline{82}$

17. Which fractions with denominator 60 have a terminating decimal?

18. For a simple repeating decimal expansion (one without a non-repeating part at the start), provide a simple description of how to immediately write it in fraction form, without using algebra.

19. Suppose a decimal expansion starts with 0.12345678910… This shows the whole numbers from 0 to 10 in order. It continues by placing every whole number in order without terminating. Is this number rational or irrational? Explain.

20. Suppose a decimal expansion starts with 0.814259668314795… After this it is non-terminating, but you don't know the rest of the digits. Can you determine if this number is rational or irrational? Explain.

21. Explain why $1.5+\sqrt{2}$ must be irrational.

22. **Challenge Problem:**

Prove that the sum of any two rational numbers is rational. Hint: Let $\frac{a}{b}$ and $\frac{c}{d}$ represent the two numbers, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$.