Enrichment Material
Irrationality of $\sqrt{2}$

How do we know that $\sqrt{2}$ is irrational? This page provides an **indirect proof** (also known as a "proof by contradiction"). In general, to give an indirect proof of "Statement A," you start with the assumption "Statement A is false." Then you proceed logically until you obtain a contradiction — something that is mathematically impossible. This implies that something that preceded the contradiction is incorrect. Because you know the validity of every step except for the original assumption, this implies that "Statement A is false" is false. In other words, this implies that Statement A is true.

Theorem

$\sqrt{2}$ is irrational.

Proof: Suppose that $\sqrt{2}$ is rational. Then there exist whole numbers m and n such that $\sqrt{2} = \frac{m}{n}$. We may assume that this fraction is reduced (if it were not reduced, then you could reduce it and use the resulting numerator and denominator). This means that m and n do not have any common factor other than 1. Square both sides of the equation:

$$\left(\sqrt{2}\right)^2 = \left(\frac{m}{n}\right)^2 \quad \text{Square.}$$

$$2 = \frac{m^2}{n^2} \quad \text{Simplify.}$$

$$2n^2 = m^2 \quad \text{Multiply both sides by the denominator.}$$

The number on the left side is a multiple of 2. This equals m^2, so m^2 is a multiple of 2. That implies that m is a multiple of 2 (m cannot be odd, because the square of an odd integer is odd). So there is another whole number q such that $m = 2q$. Substitute this into the last equation:

$$2n^2 = (2q)^2 \quad \text{Substitute.}$$

$$2n^2 = 4q^2 \quad \text{Simplify.}$$

$$n^2 = 2q^2 \quad \text{Divide both sides by 2.}$$

Using the same logic as above, it follows from this last equation that n is a multiple of 2. So we have deduced that both m and n are multiples of 2. However, this contradicts the fact that m and n do not have any common factor other than 1. This contradiction implies that the original assumption is false, therefore $\sqrt{2}$ is irrational.■

Barbara Crook 9/2/2020 12:47 PM
Comment [1]: What's up with this?

It is also true that $\sqrt{n}$ is irrational for any whole number *n* that is not a perfect square.

Exercise: You know that $\sqrt{4}$ *is* rational. Pretend that you do not and try to apply the reasoning used above to prove it is irrational. Explain where the proof "breaks down."